

PARTIAL ORTHOGONAL SPREADS OVER \mathbb{F}_2 INVARIANT UNDER THE SYMMETRIC AND ALTERNATING GROUPS

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ABSTRACT. Let $m \geq 3$ be an integer and let V be a vector space of dimension 2^m over \mathbb{F}_2 . Let Q be a non-degenerate quadratic form of maximal Witt index 2^{m-1} defined on V . We show that the symmetric group Σ_{2m+1} acts on V as a group of isometries of Q and leaves invariant a partial orthogonal spread of size $2m+1$. This is enough to show that any group of even order $2m$ or odd order $2m+1$, $m \geq 3$, acts transitively and regularly on a partial orthogonal spread in V . We construct the Σ_{2m+1} linear action on V by means of the spin representation of the symmetric group. Furthermore, when $m \equiv 3 \pmod{4}$, we show that the partial spread can be extended by two further maximal totally singular subspaces that are interchanged by Σ_{2m+1} . We also show that the alternating group \mathcal{A}_9 acts in a natural manner on a complete spread of size 9 defined on a vector space of dimension 8 over \mathbb{F}_2 .

1. INTRODUCTION

Let K be a finite field of characteristic 2, with $|K| = q$. Let V be a vector space of even dimension $2r$ over K . A quadratic form defined on V is a function $Q : V \rightarrow K$ that satisfies

$$\begin{aligned} Q(\lambda u) &= \lambda^2 Q(u) \\ Q(u+v) &= Q(u) + Q(v) + f(u, v) \end{aligned}$$

for all λ in K and all u and v in V . In this formula, f is an alternating bilinear form, called the polarization of Q . We say that Q is a non-degenerate quadratic form if f is a non-degenerate bilinear form.

We say that a vector $v \in V$ is singular with respect to Q if $Q(v) = 0$. We say that a subspace U of V is totally isotropic with respect to f if $f(u, w) = 0$ for all u and w in U . We also say that U is totally singular with respect to Q if all the elements of U are singular. It should be clear that if U is totally singular with respect to Q , then it is totally isotropic with respect to f , but the converse is not necessarily true.

We shall assume henceforth that Q is non-degenerate. In this case, the maximum dimension of a totally singular subspace of V is r . We say that Q has index r if there is at least one r -dimensional totally singular subspace of V . We shall also assume henceforth that Q has index r .

We recall that an isometry of Q is a K -linear automorphism, σ , say, of V that satisfies

$$Q(\sigma v) = Q(v)$$

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for all $v \in V$. The set of all isometries of Q forms a group under composition, called the orthogonal group of Q . This group is determined up to isomorphism by stipulating, as we have done, that Q is non-degenerate and has index r . We denote this orthogonal group by $O_{2r}^+(K)$, or $O_{2r}^+(q)$.

We call a set of r -dimensional totally singular subspaces of V a *partial orthogonal spread* for V and Q if the subspaces intersect trivially in pairs. Since V contains precisely

$$(q^{r-1} + 1)(q^r - 1)$$

non-zero singular vectors, it follows that a partial orthogonal spread contains at most $q^{r-1} + 1$ subspaces. We refer to a partial orthogonal spread of size $q^{r-1} + 1$ as a complete orthogonal spread.

It is a non-trivial fact that V contains a complete orthogonal spread if and only if r is even. On the other hand, if r is odd, the maximum size of a partial orthogonal spread in V is two.

It is clear that if σ is an element of $O_{2r}^+(K)$, and U is a totally singular subspace of V , $\sigma(U)$ is also a totally singular subspace. Thus, $O_{2r}^+(K)$ permutes the maximal (r -dimensional) totally singular subspaces, and it is known that this permutation action is transitive. It is a remarkable fact, proved by Kantor and Williams, that provided r is even, V has a complete orthogonal spread that consists of a regular orbit of a single maximal totally singular subspace under the action of a cyclic group of order $q^{r-1} + 1$. See Theorem 6.5 and the proof of Theorem 3.3 (i) in [4].

The main purpose of this paper is to construct a partial orthogonal spread of size $2m + 1$ in V invariant under the action of the symmetric group Σ_{2m+1} , when $\dim V = 2^m$, $m \geq 3$ is an integer, and $K = \mathbb{F}_2$. This enables us to prove the following result. Let G be a finite group with $|G| = 2m + \epsilon$, where $\epsilon = 0$ or 1 . Then if $m \geq 3$, G acts transitively and regularly on a partial orthogonal spread in V . We also show that the alternating group \mathcal{A}_9 acts in a natural manner on a complete spread of size 9 defined on a vector space of dimension 8 over \mathbb{F}_2 .

2. MODULES FOR Σ_n IN CHARACTERISTIC 2

The irreducible representations of the symmetric group Σ_n over a field of characteristic 2 may all be realized in \mathbb{F}_2 . They are labelled by partitions of n where no two parts are equal (so-called 2-regular partitions). We let D^λ denote the irreducible $\mathbb{F}_2\Sigma_n$ -module labelled by the 2-regular partition λ .

There is one partition λ of n for which the corresponding module D^λ has remarkable properties, which we wish to describe. Suppose that $n \geq 3$ is odd, and set $n = 2m + 1$. Consider the partition $(m + 1, m)$ of n . It is known that

$$\dim D^{(m+1,m)} = 2^m.$$

Similarly, suppose that $n \geq 4$ is even and write $n = 2m$. Then

$$\dim D^{(m+1,m-1)} = 2^{m-1}$$

in this case.

Given an $\mathbb{F}_2\Sigma_n$ -module, D , say, let $D \downarrow_{\Sigma_{n-1}}$ denote its restriction to Σ_{n-1} . From what we have described, $D \downarrow_{\Sigma_{n-1}}$ has a composition series consisting of irreducible modules of the form D^μ , where μ is a 2-regular partition of $n - 1$.

When $n = 2m + 1$ is odd, $D^{(m+1,m)} \downarrow_{\Sigma_{2m}}$ is indecomposable and has a composition series consisting of two copies of $D^{(m+1,m-1)}$. (When $m = 1$, $D^{(m+1,m-1)} =$

$D^{(2)}$ is the trivial module of Σ_2 .) When $n = 2m$ is even,

$$D^{(m+1,m-1)} \downarrow_{\Sigma_{2m-1}} = D^{(m,m-1)}$$

is irreducible. These results are proved in the main theorem of [7].

We shall call these irreducible modules $D^{(m+1,m)}$ and $D^{(m+1,m-1)}$ spin modules for Σ_{2m+1} , Σ_{2m} , respectively, over \mathbb{F}_2 .

We say that an irreducible $\mathbb{F}_2\Sigma_n$ -module D is of quadratic type if there is a non-degenerate Σ_n -invariant quadratic form Q , say, defined on D (thus Σ_n acts as isometries of Q).

We do not know necessary and sufficient conditions on a 2-regular partition λ that tell us if D^λ is of quadratic type or not, except in the case that λ has only two parts. This enables us to show that the spin modules are of quadratic type provided $n \geq 7$.

We quote the following lemma, proved in [5], Theorem 1.

Lemma 1. *Let λ be a 2-regular partition of n into two parts. Then D^λ is not of quadratic type precisely when the smaller part of λ is a power of 2, say 2^r , where $r \geq 0$, and $n \equiv k \pmod{2^{r+2}}$, where k is one of the 2^r consecutive integers $2^{r+1} + 2^r - 1, \dots, 2^{r+2} - 2$.*

We can now deal with the quadratic type of the spin module.

Lemma 2. *The spin module of Σ_n over \mathbb{F}_2 is of quadratic type except when $n = 5$ or 6.*

Proof. Suppose that $n = 2m$ is even and the spin module $D^{(m+1,m-1)}$ is not of quadratic type. Then Lemma 1 implies that $m - 1 = 2^r$, for some integer $r \geq 0$, and hence $n = 2^{r+1} + 2$. We must also have

$$n \equiv k \pmod{2^{r+2}},$$

where k is one of $2^{r+1} + 2^r - 1, \dots, 2^{r+2} - 2$.

It follows certainly that 2^{r+1} divides $k - 2$, but 2^{r+2} does not divide $k - 2$. Clearly, then, k cannot equal 2, nor can it equal $2^{r+2} + 2$, and the possibility that $k - 2 \geq 3 \times 2^{r+1}$ is excluded by the inequality $k \leq 2^{r+2} - 2$. Thus we must have $k = n = 2 + 2^{r+1}$.

When we take account of the inequality

$$2^{r+1} + 2^r - 1 \leq k = 2^{r+1} + 2,$$

we deduce that $2^r \leq 3$ and hence $r = 0$ or 1.

It is impossible that $r = 0$, since it implies $k = 4$, which is inconsistent with the inequality $k \leq 2^{r+2} - 2 = 2$. Thus, $r = 1$, $k = 6$, and it is the case that $D^{(4,2)}$ is not of quadratic type. This deals with the case that n is even.

Suppose next that $n = 2m + 1$ is odd. We have noted above that

$$D^{(m+2,m)} \downarrow_{\Sigma_{2m+1}} = D^{(m+1,m)}.$$

Since $D^{(m+2,m)}$ is of quadratic type for $m \geq 3$ by what we have proved above, we deduce that $D^{(m+1,m)}$ is also of quadratic type for $m \geq 3$. However, $D^{(3,2)}$ is not of quadratic type, by the criterion of Lemma 1. This completes the analysis. \square

Lemma 3. *Let Q be a non-degenerate Σ_{2m+1} -invariant quadratic form defined on the spin module $D^{(m+1,m)}$. Let U be the socle of $D^{(m+1,m)} \downarrow_{\Sigma_{2m}}$. Then, U is irreducible, has dimension 2^{m-1} and is totally singular with respect to Q .*

Proof. We have remarked that $D^{(m+1,m)} \downarrow_{\Sigma_{2m}}$ is indecomposable, having exactly two composition factors, both isomorphic to $D^{(m+1,m-1)}$. It follows that the socle U of $D^{(m+1,m)} \downarrow_{\Sigma_{2m}}$ is irreducible, isomorphic to $D^{(m+1,m-1)}$, and hence has dimension 2^{m-1} .

Let f be the polarization of Q and let U^\perp be the perpendicular subspace of U with respect to f . Since U and f are invariant under Σ_{2m} , U^\perp is also invariant under the group. It follows that $U \cap U^\perp$ is a subspace of U that is also Σ_{2m} -invariant. The irreducibility of U implies that $U \cap U^\perp = 0$ or U .

Now if $U \cap U^\perp = 0$, the general equality that

$$\dim U + \dim U^\perp = \dim D^{(m+1,m)}$$

implies that $D^{(m+1,m)} \downarrow_{\Sigma_{2m}}$ is the direct sum of the two invariant submodules U and U^\perp . This contradicts the fact that U is the socle.

We deduce that $U \cap U^\perp = U$, and hence U is totally isotropic with respect to f . This implies that

$$Q(u + w) = Q(u) + Q(w)$$

for all elements u and w in U , and hence Q is a linear mapping from U to \mathbb{F}_2 . The subset of all singular vectors in U is then a subspace of U of codimension at most one in U , and this subspace is certainly Σ_{2m} -invariant. The irreducibility of U forces the conclusion that Q is identically zero on U , and hence U is totally singular. □

3. CONSTRUCTION OF AN INVARIANT PARTIAL ORTHOGONAL SPREAD

We have compiled enough information about the spin module to prove the following result relating to a partial orthogonal spread invariant under symmetric group action.

Theorem 1. *Suppose that $m \geq 3$. Let V be a vector space of dimension 2^m over \mathbb{F}_2 that defines the spin module for Σ_{2m+1} and let U denote the socle of $V \downarrow_{\Sigma_{2m}}$. Let $g_1 = 1, \dots, g_{2m+1}$ be a set of coset representatives for Σ_{2m} in Σ_{2m+1} . Then the $2m+1$ subspaces $U = g_1U, g_2U, \dots, g_{2m+1}U$ form a partial orthogonal spread in V , permuted by Σ_{2m+1} according to its natural action.*

Proof. We first note that, since $m \geq 3$, V is indeed a module of quadratic type, by Lemma 2, and Lemma 3 implies that U is totally singular with respect to the invariant non-degenerate quadratic form. Let g be any element of Σ_{2m+1} not in Σ_{2m} . Consider the subspace gU , which is also totally singular.

We claim that $gU \neq U$. For if $gU = U$, U is invariant under the subgroup of Σ_{2m+1} generated by g and Σ_{2m} . Since Σ_{2m} is a maximal subgroup of Σ_{2m+1} , this subgroup is all of Σ_{2m+1} . But as V is irreducible for Σ_{2m+1} , U cannot be invariant under Σ_{2m+1} . We deduce that $gU \neq U$, as claimed.

Consider now the subspace $U \cap gU$, which we have just shown is not the whole of U . It is easy to see that $U \cap gU$ is invariant under the subgroup $\Sigma_{2m} \cap (g\Sigma_{2m}g^{-1})$ of Σ_{2m} . If we assume, as we may, that Σ_{2m} is the subgroup of Σ_{2m+1} fixing 1 in the natural representation of Σ_{2m+1} on the numbers $\{1, 2, \dots, 2m+1\}$, we see that

$\Sigma_{2m} \cap (g\Sigma_{2m}g^{-1})$ is the subgroup fixing 1 and $g(1)$. We may then identify this subgroup unambiguously as Σ_{2m-1} , since the subgroups of Σ_{2m+1} fixing 1 and a different number are conjugate.

We know that U affords the spin representation of Σ_{2m} , and its restriction to Σ_{2m-1} is irreducible. Thus the only subspaces of U that are invariant under Σ_{2m-1} are U and 0. We deduce that $U \cap gU = 0$, as required, and the rest of the theorem follows from this argument. \square

Corollary 1. *Let G be a finite group of order at least 6. Then G acts in a regular transitive manner on a partial orthogonal spread of size $|G|$ defined on a quadratic space of dimension 2^m over \mathbb{F}_2 , where m is the integer part of $|G|/2$.*

Proof. Suppose first that $|G| = 2m + 1$, where $m \geq 3$. We may embed G into Σ_{2m+1} by means of its regular representation. Then G permutes the subspaces in the partial orthogonal spread described in Theorem 1 in a regular transitive manner, as required.

Suppose next that $|G| = 2m$ is even, with $m \geq 3$. We may embed G into Σ_{2m+1} in such a way that G fixes one point and permutes the remaining $2m$ points regularly. Then, in the action on the partial orthogonal spread described in Theorem 1, G clearly fixes one subspace and permutes the other $2m$ subspaces regularly. \square

Note. If $|G| = 3, 4$ or 5 , we may embed G into Σ_7 and then show that G acts transitively on a partial orthogonal spread of size $|G|$ defined on a quadratic space of dimension 8 over \mathbb{F}_2 . However, a group of order 4 or 5 does not act transitively on a partial orthogonal spread of size 4 or 5 on a space of dimension 4 over \mathbb{F}_2 , since a complete spread only contains three subspaces in such a case. A similar remark holds for a group of order 3 acting on a two-dimensional space over \mathbb{F}_2 .

4. EXTENSION OF THE PARTIAL SPREAD WHEN $m \equiv 3 \pmod{4}$

When we take into account the influence of the alternating subgroup \mathcal{A}_{2m+1} of Σ_{2m+1} , we shall show that if $m \equiv 3 \pmod{4}$, the partial orthogonal spread of size $2m + 1$ just described can be extended by two more maximal totally singular subspaces to give a partial orthogonal spread of size $2m + 3$. The two additional subspaces are invariant under the alternating group \mathcal{A}_{2m+1} and are interchanged by any odd permutation in Σ_{2m+1} .

In order to find these additional subspaces, it is necessary to describe how \mathcal{A}_{2m+1} acts on the spin module. The results we need to know are quite sensitive to properties of the integer m , and require careful explanation.

Lemma 4. *The spin module $D^{(m+1,m)}$ of Σ_{2m+1} over \mathbb{F}_2 splits as a direct sum of two non-isomorphic irreducible $\mathbb{F}_2\mathcal{A}_{2m+1}$ -modules if $m \equiv 0 \pmod{4}$ or if $m \equiv 3 \pmod{4}$. The two \mathcal{A}_{2m+1} -modules are conjugate under the action of Σ_{2m+1} , as described by Clifford's theorem.*

Proof. This follows from Theorem 6.1 of [1]. \square

The next question we need to address is whether or not the two irreducible $\mathbb{F}_2\mathcal{A}_{2m+1}$ -modules described in Lemma 4 are self-dual. Here again, the answer is not obvious, and depends on the residue of m modulo 4. It seems that to establish what we want to know, we must invoke an alternative construction of the spin module over \mathbb{F}_2 .

Let Γ_{2m+1} denote either of the two non-isomorphic double covers of Σ_{2m+1} . The commutator subgroup of Γ_{2m+1} has index 2 in Γ_{2m+1} and is a double cover of \mathcal{A}_{2m+1} , which we shall denote by $\tilde{\mathcal{A}}_{2m+1}$. Since $\tilde{\mathcal{A}}_{2m+1}$ is an extension of a central subgroup of order 2 by \mathcal{A}_{2m+1} , given any element of odd order in \mathcal{A}_{2m+1} , there is a unique element of the same order that projects onto it under the canonical homomorphism from $\tilde{\mathcal{A}}_{2m+1}$ onto \mathcal{A}_{2m+1} . We shall refer to this element of $\tilde{\mathcal{A}}_{2m+1}$ as the canonical inverse image of the given element of odd order in \mathcal{A}_{2m+1} .

Γ_{2m+1} has a faithful irreducible complex representation of degree 2^m , known as the basic spin representation (it is an example of a so-called projective representation of Σ_{2m+1}). Let θ denote the character of the basic spin representation. Schur shows in [6], Formula VII*, p. 205, that θ is rational-valued. Furthermore, it follows Theorem 7.7 of [8] that θ defines an absolutely irreducible Brauer character modulo the prime 2. Corollary 9.4 of Chapter IV of [2] implies then that θ has Schur index one over the field \mathbb{Q}_2 of 2-adic numbers. Thus, since θ certainly takes values in \mathbb{Q}_2 , we deduce that the basic spin representation may be realized over \mathbb{Q}_2 .

Let \mathbb{Z}_2 denote the ring of 2-adic integers in \mathbb{Q}_2 . R is a principal ideal domain and it follows that there is a Γ_{2m+1} -invariant \mathbb{Z}_2 -lattice L , say, of rank 2^m which affords the basic spin representation. The quotient $L/2L$ is then a vector space, \bar{L} , say, of dimension 2^m over \mathbb{F}_2 . Since the central involution of Γ_{2m+1} acts as $-I$ on L , this involution acts trivially on \bar{L} , and thus \bar{L} is naturally an $\mathbb{F}_2\Sigma_{2m+1}$ -module, which it turns out is isomorphic to the spin module $D^{(m+1,m)}$ we have been considering.

Working over the algebraic closure of \mathbb{Q}_2 , the basic spin module is reducible on restriction to $\tilde{\mathcal{A}}_{2m+1}$. This splitting does not necessarily occur over \mathbb{Q}_2 , since we need a square root of $(-1)^m(2m+1)$ for it to take place. We refer to [6], Formula VII*, p. 205, for this theory. Schur shows that θ splits into two different irreducible characters of $\tilde{\mathcal{A}}_{2m+1}$, θ_1 and θ_2 , say. These characters θ_1 and θ_2 are real-valued if and only if m is even. Furthermore, θ_1 and θ_2 differ on the canonical inverse image of a $2m+1$ -cycle. In particular, if m is odd, θ_1 and θ_2 take non-real values on the canonical inverse image of a $2m+1$ -cycle.

Now θ restricted to elements of odd order is the Brauer character of the spin module $D^{(m+1,m)}$ of Σ_{2m+1} . Since we know that $D^{(m+1,m)}$ is reducible on restriction to \mathcal{A}_{2m+1} , the Brauer characters of the irreducible constituents are θ_1 and θ_2 , again restricted to elements of odd order. Finally, since θ_1 and θ_2 are not real-valued on the canonical inverse image of a $2m+1$ -cycle, the Brauer characters defined by θ_1 and θ_2 are not real-valued, and consequently, the two irreducible constituents of $D^{(m+1,m)} \downarrow_{\mathcal{A}_{2m+1}}$ are not self-dual.

When we collect the information we have derived from the work of Schur, we have proved the following important result.

Lemma 5. *The spin module $D^{m+1,m}$ of Σ_{2m+1} over \mathbb{F}_2 splits as a direct sum of two non-isomorphic irreducible $\mathbb{F}_2\mathcal{A}_{2m+1}$ -modules if $m \equiv 3 \pmod{4}$. The two $\mathbb{F}_2\mathcal{A}_{2m+1}$ -modules are not self-dual in this case.*

We proceed to extend the partial orthogonal spread when $m \equiv 3 \pmod{4}$.

Theorem 2. *Suppose that $m \equiv 3 \pmod{4}$. Let V be a vector space of dimension 2^m over \mathbb{F}_2 that defines the spin module for Σ_{2m+1} and let U denote the socle of $V \downarrow_{\Sigma_{2m}}$. Let $V \downarrow_{\mathcal{A}_{2m+1}} = U_1 \oplus U_2$, where U_1 and U_2 are non-isomorphic irreducible $\mathbb{F}_2\mathcal{A}_{2m+1}$ -modules. Then U_1 and U_2 are both totally singular.*

Furthermore, let $g_1 = 1, \dots, g_{2m+1}$ be a set of coset representatives for Σ_{2m} in Σ_{2m+1} . Then $U = g_1U, g_2U, \dots, g_{2m+1}U, U_1$ and U_2 form a partial orthogonal spread in V consisting of $2m + 3$ subspaces.

Proof. We note that all $2m + 3$ subspaces described above have dimension 2^{m-1} . We also proved in Lemma 5 that U_1 and U_2 are not self-dual. Since they are both irreducible under the action of \mathcal{A}_{2m+1} , it follows in a straightforward way (as in the proof of Lemma 3) that U_1 and U_2 are both totally singular.

We claim that $U \neq U_1$. For, U is invariant under Σ_{2m} , whereas U_1 is invariant under \mathcal{A}_{2m+1} but not under Σ_{2m+1} . Let g be any odd permutation in Σ_{2m} . Then $g \notin \mathcal{A}_{2m+1}$ also, and hence $gU_1 = U_2$. Now if $U = U_1$, then $U = gU = gU_1 = U_2$. This is clearly a contradiction.

We deduce that $U \cap U_1$ is a proper subspace of U , since $\dim U = \dim U_1$. $U \cap U_1$ is also invariant under $\Sigma_{2m} \cap \mathcal{A}_{2m+1} = \mathcal{A}_{2m}$. We also know from the previous section of this paper that U is an irreducible $\mathbb{F}_2\Sigma_{2m}$ -module, isomorphic to the spin module $D^{(m+1, m-1)}$. Since m is odd by assumption, $D^{(m+1, m-1)} \downarrow_{\mathcal{A}_{2m}}$ is irreducible, by Theorem 1.1 of [1]. Thus the only \mathcal{A}_{2m} -submodules of U are U and 0. Since $U \cap U_1$ is an \mathcal{A}_{2m} -submodule, and not equal to U , it must be 0. A similar argument proves that $U \cap U_2 = 0$ also.

Finally, let h be any element of Σ_{2m+1} not in Σ_{2m} , and consider $(hU) \cap U_1$. Since $h^{-1}U_1 = U_1$ or U_2 , we see that

$$(hU) \cap U_1 = h(U \cap h^{-1}U_1) = 0,$$

since we have proved that $U \cap U_1 = U \cap U_2 = 0$. Similarly, $(hU) \cap U_2 = 0$ also. This completes the proof. \square

This argument shows, for example, that the partial orthogonal spread of seven subspaces in an 8-dimensional space over \mathbb{F}_2 , invariant under the action of Σ_7 , can be extended to a complete spread of nine subspaces, also invariant under Σ_7 .

5. ACTION OF \mathcal{A}_9 ON A COMPLETE SPREAD IN 8 DIMENSIONS

We have just shown that Σ_7 acts on a complete spread of nine subspaces in an 8-dimensional space over \mathbb{F}_2 . We intend to give another explanation of this fact by showing that \mathcal{A}_9 acts on a complete spread of nine subspaces in an 8-dimensional space over \mathbb{F}_2 and then observing that Σ_7 is a subgroup of \mathcal{A}_9 .

We take as our starting point the data that \mathcal{A}_9 has three inequivalent irreducible representations of degree 8 over \mathbb{F}_2 . We need to exclude one of these representations from consideration, and this is the so-called deleted permutation module, which arises from the natural permutation action of \mathcal{A}_9 on nine points. The restriction of the deleted permutation module to \mathcal{A}_8 has a composition series consisting of an irreducible module of dimension 6 and two copies of the trivial module, and this is not what we want.

The modules that we require arise from the restriction of the 16-dimensional spin module $D^{(5,4)}$ of Σ_9 to \mathcal{A}_9 . We remarked already that Theorem 6.1 of [1] implies that $D^{(5,4)} \downarrow_{\mathcal{A}_9}$ is the direct sum of two non-isomorphic $\mathbb{F}_2\mathcal{A}_9$ -modules. These two 8-dimensional modules are both self-dual. By way of proof, albeit not a self-contained one, we can refer to p.85 of [3], where we see that all three inequivalent irreducible representations of \mathcal{A}_9 of degree 8 in characteristic 2 support \mathcal{A}_9 -invariant quadratic forms.

We can now proceed to fashion this information into a statement about the action of \mathcal{A}_9 on a complete orthogonal spread.

Theorem 3. *Let V be an $\mathbb{F}_2\mathcal{A}_9$ -module of dimension 8 defined as an irreducible constituent of the restriction of the spin module $D^{(5,4)}$ of Σ_9 . Then V is a module of quadratic type. Let U be an irreducible $\mathbb{F}_2\mathcal{A}_8$ -submodule of V . Then U is 4-dimensional and is totally singular. Moreover, if g_1, \dots, g_9 are a set of coset representatives for \mathcal{A}_8 in \mathcal{A}_9 , the nine subspaces g_iU , $1 \leq i \leq 9$, form a complete orthogonal spread in V .*

Proof. We first note that we can take the Brauer character of \mathcal{A}_9 acting on V to be that denoted by ϕ_3 in the table found on p.85 of [3]. (The character ϕ_4 has the same properties as ϕ_3 , and is conjugate to ϕ_3 under the action of Σ_9 .) As we noted above, V is of quadratic type. (This also follows from the fact that $D^{(5,4)}$ is of quadratic type and V is self-dual, since its Brauer character is real-valued.)

Reference to the table on p.48 of [3] shows that the restriction of ϕ_3 to \mathcal{A}_8 consists of two different irreducible Brauer characters of degree 4, one being the complex conjugate of the other. Now it is a fact that V is reducible as an $\mathbb{F}_2\mathcal{A}_8$ -module. To see this, we note that \mathcal{A}_8 is isomorphic to the general linear group $\text{GL}_4(\mathbb{F}_2)$. The Brauer characters that occur in the restriction of ϕ_3 to \mathcal{A}_8 are those of the natural 4-dimensional module for $\text{GL}_4(\mathbb{F}_2)$ over \mathbb{F}_2 and its contragredient (or dual). If V were irreducible as an $\mathbb{F}_2\mathcal{A}_8$ -module, it would follow that the two non-isomorphic irreducible modules for $\text{GL}_4(\mathbb{F}_2)$ were Galois-conjugate over \mathbb{F}_4 , which is not the case, as they are defined over \mathbb{F}_2 .

This argument establishes that U is 4-dimensional and furthermore, it is not self-dual, since its Brauer character is not real-valued. This then implies that U is totally singular.

To complete the proof, we imitate the proof of Theorem 1.

Let g be any element of \mathcal{A}_9 not in \mathcal{A}_8 . Consider the subspace gU , which is also totally singular. We claim that $gU \neq U$. For if $gU = U$, U is invariant under the subgroup of \mathcal{A}_9 generated by g and \mathcal{A}_8 . Since \mathcal{A}_8 is a maximal subgroup of \mathcal{A}_9 , this subgroup is all of \mathcal{A}_9 . But as V is irreducible for \mathcal{A}_9 , U cannot be invariant under \mathcal{A}_9 . We deduce that $gU \neq U$, as claimed.

Consider now the subspace $U \cap gU$, which we have just shown is not the whole of U . It is easy to see that $U \cap gU$ is invariant under the subgroup $\mathcal{A}_8 \cap (g\mathcal{A}_8g^{-1})$ of \mathcal{A}_9 . We may take \mathcal{A}_8 to be the subgroup of \mathcal{A}_9 fixing 1 in the natural representation of \mathcal{A}_9 on the numbers $\{1, 2, \dots, 9\}$. Then we see that $\mathcal{A}_8 \cap (g\mathcal{A}_8g^{-1})$ is the subgroup fixing 1 and $g(1)$, which is isomorphic to \mathcal{A}_7 .

We know that U affords an irreducible representation of \mathcal{A}_8 , and its restriction to \mathcal{A}_7 is irreducible. Thus the only subspaces of U that are invariant under \mathcal{A}_7 are U and 0. We deduce that $U \cap gU = 0$, as required, and the rest of the theorem follows from this argument. \square

We note that any group G of order 9 may be embedded into \mathcal{A}_9 by its regular representation. It follows that G acts in a regular transitive manner on a complete orthogonal spread of size 9 defined on a quadratic space of dimension 8 over \mathbb{F}_2 . This is a consequence of the theorem of Kantor and Williams already described when G is cyclic, but seems to be a new observation when G is elementary abelian. Similarly, any non-cyclic group G of order 8 may be embedded into \mathcal{A}_8 by its regular representation, and then into \mathcal{A}_9 . It follows that G acts in a regular transitive

manner on a partial orthogonal spread of size 8 defined on a quadratic space of dimension 8 over \mathbb{F}_2 .

There are 135 non-zero singular vectors in the 8-dimensional quadratic space of index 4 over \mathbb{F}_2 . These vectors are permuted transitively by \mathcal{A}_9 . The action is imprimitive, there being nine blocks of imprimitivity, namely, the non-zero vectors in each of the 4-dimensional subspaces that constitute the invariant complete spread. The stabilizer of a block is isomorphic to \mathcal{A}_8 , and it acts doubly transitively on the 15 non-zero vectors in the block.

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